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GRAPHICAL AND ANALYTICAL METHODS FOR THE DETERMINATION OF A FLOW OF A COMPRESSIBLE FLUID AROUND AN OBSTACLE

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SUMMARY

Chaplygin introduced the hodograph method in the theory of compressible fluid flows and developed a method for ~~con-~~ constructing stream functions of such flows. This method, which has been extensively used in investigation of compressible fluid flows, is limited in certain respects. The expression for the stream function obtained in this manner can represent only certain types of flow patterns. In general, flow patterns obtained in this way cannot represent the whole flow around an obstacle, but only a part of such a flow, and therefore several expressions are needed in order to obtain the whole flow. On the other hand, in many instances, it is important to have a single expression representing the whole flow.

Recently Von Kármán and Tsien constructed more general types of stream functions, but only by replacing the true pressure density relation

$$p = \sigma\rho^k$$

by the linear pressure-specific volume relation

$$p = A + \frac{\sigma}{\rho}$$

(A, k, and σ are constants), so that their method is essentially limited to flows the maximum Mach number of which is not too large.

In a companion report the author derived a new formula for stream functions based on the true pressure density relation

$$p = \sigma\rho^k$$

It is not subject to the limitations of the Chaplygin method.

In the present report this formula is employed to construct two-dimensional subsonic compressible fluid flows around a body similar in shape to a given symmetric obstacle.

The methods described in the report are illustrated by numerical examples.

INTRODUCTION

In a companion report (reference 1) the author derived a formula transforming an arbitrary analytic function of a complex variable, $g(Z)$, into a function $\psi(x,y)$ which satisfies the differential equation for the stream function of a two-dimensional potential gas flow. (It is assumed that the pressure of the fluid is a function of its density.)

This report describes the application of this formula to the determination of a flow around an obstacle. First, the actual computation of the stream function generated by an analytic function is described in detail. Then various methods for choosing the function which yields a flow around a given obstacle (or at least around a shape similar to the given one) are discussed.

The procedure is illustrated by completely carrying out the construction of the compressible flow generated by the function

$$g(Z) = \frac{1}{2}\left(\frac{1}{\sqrt{1 - 2e^Z}} + \sqrt{1 - 2e^Z}\right)$$

In the case of an incompressible fluid this function would lead to the circulation-free flow around a circular cylinder. In the case of a compressible fluid a flow around a cylinder of somewhat distorted cross section is obtained.

For the mathematical justification of the formulas used, the reader is referred to reference 1.

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I take the opportunity to express my gratitude to Mr. Leonard Greenstone for his assistance in the preparation of the present paper.

NOTATION

a speed of sound

a_o speed of sound at a stagnation point

$f(z)$ analytic function of the complex variable z

$g(Z)$ analytic function of the complex variable Z

$g^{(n+1)}(Z)$ nth iterated integral of $g(Z)$ (See (16).)

H a function of a real variable occurring in the formula for the stream function of a compressible flow (See sec. 2.)

k exponent in the pressure-density relation

M Mach number $(M = v/a)$

$Q^{(n)}$ function of a real variable occurring in the formula for the stream function of a compressible flow

T_n imaginary part of $g^{(n)}$ considered as a function of M and θ

v speed (magnitude of the velocity vector)

v_1, v_2 components of the velocity vector

$w = \varphi + i\psi$ complex potential

$z = x + iy$ complex variable in the physical plane

$Z = \lambda - i\theta$ complex variable in the logarithmic plane

θ angle between the velocity vector and the x-axis

λ for an incompressible flow: logarithm of the speed; for a compressible flow: a given function of the local Mach number (defined in sec. 2) (The λ of this report is not to be confused with λ of reference 1, to which it is similar, but differs from it by a constant.)

φ velocity potential

φ_n real part of $g^{(n)}$ (See (20).)

ψ stream function

ψ_n imaginary part of $g^{(n)}$ (See (20).)

1. Construction of a Flow by Means of Analytic Functions of a Complex Variable (Incompressible Fluid)

The method used in this paper consists of generating two-dimensional compressible flows by means of analytic functions of a complex variable. The method is best explained by first considering the case of an incompressible fluid.

There exists a very simple method of constructing patterns of a two-dimensional steady irrotational flow of an incompressible perfect fluid by means of analytic functions of a complex variable. Let

$$w = f(z) = \varphi(x,y) + i\psi(x,y), \quad z = x + iy \tag{1}$$

be such a function. Then φ and ψ satisfy the Laplace equation,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

where φ can be interpreted as the velocity potential of an incompressible flow; ψ is the stream function, and

$$\psi(x,y) = \text{constant}$$

is the equation of a streamline.

For instance,

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{x(x^2 + y^2 + 1)}{2(x^2 + y^2)} + i\,\frac{(x^2 + y^2 - 1)}{2(x^2 + y^2)} \qquad (2)$$

represents a circulation-free flow around a circular cylinder, for $\operatorname{Im}(w)$ is constant for $|z| = 1$. (See fig. 1.)

Unfortunately, this simple method cannot be extended to the case of a compressible fluid.

Now, let v_1 and v_2 be the velocity components of the incompressible flow

$$v_1 = \partial\varphi/\partial x, \quad v_2 = \partial\varphi/\partial y$$

Let v be the speed and θ the angle between the velocity vector and the x-axis. Then

$$v_1 - iv_2 = ve^{-i\theta} = \frac{dw}{dz} \qquad (3)$$

Thus $v_1 - iv_2$ is also an analytic function of z. By virtue of (3) to each point (x,y) of the physical plane, there corresponds a point $(v_1, -v_2)$ of the plane the Cartesian coordinates of which are $v_1, -v_2$ (hodograph plane). In this way an image of the flow in the $(v_1, -v_2)$-plane is obtained. This image is called the hodograph of the flow. It may happen that at two different points of the (x,y)-plane (physical plane) v_1 and v_2 have the same value. Then, to these two points there corresponds the same point in the hodograph plane. If it is desirable to have a one-to-one correspondence between the flow and its hodograph, the latter must be interpreted as a "Riemann surface." (See reference 2, pp. 27, 59-60, etc.)

For the flow (2)

$$\frac{dw}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

The hodograph of the flow consists of all points within the circle $\left(v_1 - \frac{1}{2}\right)^2 + v_2{}^2 = \frac{1}{4}$. Each point of the hodograph corresponds to two points in the z-plane. (See fig. 2.)

In the hodograph plane v and $-\theta$ are polar coordinates.

The "logarithmic" plane is defined as the plane in which λ,

$$\lambda = \log v$$

and $-\theta$ are the Cartesian coordinates. The complex variable in the logarithmic plane is

$$Z = \lambda - i\theta = \log\left(\frac{dw}{dz}\right) \tag{4}$$

The transformation (4) maps the hodograph of the flow into its image in the logarithmic plane. Note that the mapping of the hodograph plane into the logarithmic plane is independent of the flow.

For instance, the image of the flow (2) in the logarithmic plane is the (doubly covered) domain bounded by the curve

$$\lambda = \log 2 + \log \cos \theta, \quad -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} \tag{5}$$

(See figs. 3 and 5.)

From the basic properties of analytic functions it follows that the complex potential w also can be considered as an analytic function of Z,

$$w = f(z) = g(Z) \tag{6}$$

For instance, for the flow (2)

$$Z = \log \frac{1}{2}\left(1 - \frac{1}{z^2}\right); \quad \text{that is,} \quad z = (1 - 2e^Z)^{-\frac{1}{2}} \tag{7}$$

and therefore

$$g(Z) = \frac{1}{2}\left[(1 - 2e^Z)^{\frac{1}{2}} + (1 - 2e^Z)^{-\frac{1}{2}}\right] \tag{8}$$

The streamlines in the physical plane (z-plane) correspond to the lines in the logarithmic plane along which

$$T_0(\lambda,\theta) = \operatorname{Im} g(Z) = \psi \tag{9}$$

is constant.

For the flow (2)

$$T_o(\lambda,\theta) = \operatorname{Im}\left[g(Z)\right]$$

$$= \frac{1}{2\sqrt{2}}\left[-(1 - 2e^{\lambda}\cos\theta) + (1 - 4e^{\lambda}\cos\theta + 4e^{2\lambda})^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\times\left[1 - \frac{1}{(1 - 4e^{\lambda}\cos\theta + 4e^{2\lambda})^{\frac{1}{2}}}\right] \qquad (10)$$

(The values of T_o are given in table III and plotted in fig. 5.) In this example the streamlines are the same in the two sheets of the Riemann surface.

Now, let an arbitrary analytic function $g(Z)$ be given. It always is possible to interpret this function as the complex potential (in the logarithmic plane) of an incompressible flow in the physical plane (z-plane). The flow in the z-plane can be determined as follows. Since $w(z) = g(Z)$, and

$$Z = \log\frac{dw}{dz} = \log\frac{dg}{dZ} + \log\frac{dZ}{dz}$$

then

$$\frac{dz}{dZ} = \frac{dg}{dZ}\,e^{-Z}$$

or

$$z = \int \frac{dg}{dZ}\,e^{-Z}\,dZ$$

Integration along a streamline (in the Z-plane) yields a streamline in the z-plane. In this case the foregoing integral can be written in the form

$$z = x + iy = -\int_0^v \frac{\cos\theta}{v^2}\,\frac{\left(\psi_\theta^{\,2} + v^2\psi_v^{\,2}\right)}{\psi_\theta}\,dv$$

$$- i\int_0^v \frac{\sin\theta}{v^2}\,\frac{\left(\psi_\theta^{\,2} + v^2\psi_v^{\,2}\right)}{\psi_\theta}\,dv$$

In order that the flow be physically possible, it is necessary that the streamlines do not intersect.

Application of this procedure to function (4) leads back to the flow (2).

By means of the foregoing method of associating with every analytic function the Z-plane (logarithmic plane), a flow in the z-plane (physical plane) can be extended to the case of a compressible flow.

2. Construction of Flows by Means of Analytic Functions of a Complex Variable (Compressible Fluid)

Consider a two-dimensional subsonic potential flow of a compressible fluid (in the z-plane). It is assumed that the density ρ and the pressure p are connected by the relation

$$p = A + \sigma \rho^k \tag{11}$$

where A, σ, and k are constants. There exists a stream function $\psi(x,y)$ such that

$$v_1 = \frac{1}{\rho}\frac{\partial \psi}{\partial y}, \quad v_2 = -\frac{1}{\rho}\frac{\partial \psi}{\partial x} \tag{12}$$

are the components of the velocity vector in the x and y directions, respectively.

Let M be the local Mach number

$$M = M(v) = v \Big/ \left[a_o^2 - \frac{1}{2}(k - 1)v^2\right]^{\frac{1}{2}} \tag{13}$$

where a_o is the speed of sound at rest, v the speed of the flow, θ the angle between the velocity vector and the positive x-axis. Set

$$2\lambda = \log\left[\left(\frac{1 - (1 - M^2)^{\frac{1}{2}}}{1 + (1 - M^2)^{\frac{1}{2}}}\right)\left(\frac{1 + h(1 - M^2)^{\frac{1}{2}}}{1 - h(1 - M^2)^{\frac{1}{2}}}\right)^{\frac{1}{h}}\right] \tag{14}$$

where

$$h = \left[\frac{k - 1}{k + 1}\right]^{\frac{1}{2}}, \quad k > 1$$

(If $k < 1$ the definition of λ must be changed. (See reference 1.)

The logarithmic plane is defined as the plane with Cartesian coordinates λ, θ. The complex variable in this plane is

$$Z = \lambda - i\theta$$

In the Z-plane ψ, which may be considered as a function of λ and θ, satisfies the equation

$$L_o(\psi) \equiv \frac{1}{4}\left(\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\theta^2}\right) + N\frac{\partial\psi}{\partial\lambda} \equiv \frac{\partial^2\psi}{\partial Z\partial\overline{Z}} + N\left(\frac{\partial\psi}{\partial Z} + \frac{\partial\psi}{\partial\overline{Z}}\right) = 0 \quad (15)$$

where

$$N = \frac{-(k + 1)M^4}{8(1 - M^2)^{3/2}}$$

This is a linear equation and its treatment can be facilitated by using results from the highly developed theory of linear partial differential equations. Let $g(Z)$ be an arbitrary analytic function of Z. Using this function, a solution of (15) can be obtained as follows:

Set

$$\left.\begin{aligned} g^{(o)}(Z) &= g(Z) \\ g^{(n+1)}(Z) &= \int_o^Z g^{(n)}(Z)dZ, \quad n = 0, 1, \ldots \end{aligned}\right\} \quad (16)$$

There exist fixed functions of a real variable ξ

$$H(\xi), \quad Q^{(n)}(\xi), \quad n = 1, 2, \ldots$$

such that the function

$$\psi\left(\frac{v}{a_o},\theta\right) = H(2\lambda - 2\alpha)\Big[\operatorname{Im} g(Z) + \frac{1}{2}Q^{(1)}(2\lambda - 2\alpha)\operatorname{Im} g^{(1)}(Z) + \frac{3}{4}Q^{(2)}(2\lambda - 2\alpha)\operatorname{Im} g^{(2)}(Z) + \ldots + \frac{(2n)!}{2^{2n}n!}Q^{(n)}(2\lambda - 2\alpha)\operatorname{Im} g^{(n)}(Z) + \ldots\Big] \tag{17}$$

(α a real constant) satisfies (15). (The representation (17) is an approximation to the solution. The exact formula proved in reference 1 is

$$\psi\left(\frac{v}{a_o},\theta\right) = H(2\lambda - 2\alpha)\lim_{m\to\infty}\Big[\operatorname{Im} g(z) + \sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}n!}Q_m^{(n)}(2\lambda - 2\alpha)\operatorname{Im} g^{(n)}(Z)\Big] \tag{18}$$

where α is an arbitrary non-negative constant

$$\lim_{m\to\infty} Q_m^{(n)} = Q^{(n)}$$

In the example under consideration it is replaced by (17), since the Mach number is comparatively small and the $Q_m^{(n)}$ do not differ considerably from $Q^{(n)}$. If, however, the local Mach number approaches the value 1, it is necessary to use the exact formula. (Another form of formula (17) is given in appendix I.)) The functions H and $Q^{(n)}$ depend only upon k and v. They have been defined by a recurrence formula and computed for n = 1, 2, 3. (See reference 1, sec.12.) Some of them are tabulated in tables Ia and Ib and plotted in diagrams Ia and Ib.

Once ψ has been determined as a function of v/a_o and θ, the transition to the physical plane (z-plane) is given by the formulas

$$\left.\begin{aligned} x &= -\int_0^v \frac{\rho_0 \cos\theta}{\rho v^2} \frac{\left(\psi_\theta^2 + v^2\psi_v^2\right)}{\psi_\theta} dv + \int_0^v \frac{\rho_0 \cos\theta \, M^2\psi_\theta}{\rho v^2} dv \\ y &= -\int_0^v \frac{\rho_0 \sin\theta}{\rho v^2} \frac{\left(\psi_\theta^2 + v^2\psi_v^2\right)}{\psi_\theta} dv + \int_0^v \frac{\rho_0 \sin\theta \, M^2\psi_\theta}{\rho v^2} dv \end{aligned}\right\} \quad (19)$$

where the integration is performed along a streamline (i.e., along a line ψ = constant) and the subscripts θ, v denote partial differentiation.

In this way an arbitrary number of streamlines in the physical plane (z-plane) can be drawn. It should be noticed that along each streamline the value of v/a_o is known and therefore also the values of the local Mach number, the pressure, and the density.

The next section contains a discussion of the actual computation of the function ψ and of the streamlines in the physical plane.

Two additional remarks may be made:

1. The series (17) converges rather slowly for values of v in the neighborhood of $v = a$ (i.e., $M = 1$). This is not due to a singularity of ψ, since it is known that ψ is regular in the domain concerned. Therefore it is possible to replace (17) by other expansions which converge more rapidly. (See reference 1, sec. 17.) The author also will treat this question in a future publication.

2. It is known that the theory becomes especially simple if $k = -1$. (The pressure-specific volume relation then becomes linear.) Therefore it is interesting to know how ψ (generated by a given analytic function $g(Z)$) changes if k varies in the interval $(-1,2)$. For this reason the computations are carried out for two values of k, $k = -0.5$, and $k = 1.4$ in the example given in the following section.

3. Actual Computation of the Stream Function, Using the Generating Operator in the Form (17)

The actual computation of a compressible flow given in the logarithmic plane by an analytic function $g(Z)$ consists of the following steps:

I. Computation of the function $g^{(n)}(Z)$

II. Construction of the flow in the logarithmic plane

III. Transition from the logarithmic plane to the physical plane

These steps are discussed in detail in the present section. The procedure is illustrated by constructing the compressible flow obtained from the function

$$g(Z) = \frac{1}{2}\left[(1 - 2e^{Z})^{-\frac{1}{2}} + (1 - 2e^{Z})^{\frac{1}{2}}\right] \tag{8}$$

(For the case of an incompressible flow this function yields the flow around a circular cylinder; see the first section.)

Step I.- The functions $g^{(n)}(Z)$ have been defined in the preceding section. In general, it will be impossible to evaluate the integrals in closed form

$$\begin{aligned} g^{(n+1)}(Z) &= \int_0^Z g^{(n)}(Z)dZ \\ &= \int_0^{\lambda+i\theta}\left[\varphi^{(n)}(\lambda,\theta) + i\psi^{(n)}(\lambda,\theta)\right](d\lambda - id\theta) \\ &= \int_0^{\lambda,\theta}(\varphi^{(n)}d\lambda + \psi^{(n)}d\theta) + i\int_0^{\lambda,\theta}(-\varphi^{(n)}d\theta + \psi^{(n)}d\lambda) \end{aligned} \tag{20}$$

where

$$\varphi_n + i\psi_n = g^{(n)}$$

however, it is possible to use numerical integration. Since $g^{(n)}(Z)$ is an analytic function of a complex variable, the value of the integral $\int_0^Z g^{(n)}(Z)dZ$ is independent of the path of integration. A very convenient path consists of the interval $[(0,0), (\lambda,0)]$ of the negative real axis, followed by the interval $[(\lambda,0), (\lambda,\theta)]$ along a line parallel to the

θ axis (provided this path lies within the image of the flow in the logarithmic plane. If this does not take place, an obvious adjustment must be made.) Dividing each of these intervals into s subintervals $\Delta\lambda_k$, $\Delta\theta_k$ gives approximately

$$g^{(n+1)}(Z) = \sum_{k=1}^{s} \varphi_k^{(n)}\Delta\lambda_k + i\sum_{k=1}^{s} \psi_k^{(n)}\Delta\lambda_k + \sum_{k=1}^{s} \psi_k^{(n)}\Delta\theta_k - i\sum_{k=1}^{s} \varphi_k^{(n)}\Delta\theta_k \qquad (21)$$

In many instances it is more convenient to expand $g(Z)$ into an infinite series and then integrate term by term. However, as a rule, function $g(Z)$ has singularities, and several series developments are needed in order to cover the domain in which the function is to be considered. For the case of function (8) integral $\int g dZ$ can be computed in closed form. Then

$$\int_0^Z g(Z)dZ = (1 - 2e^Z)^{\frac{1}{2}} + \log\left[e^Z - 1 + (1 - 2e^Z)^{\frac{1}{2}}\right] - Z - i\left(\frac{2+\pi}{2}\right) \qquad (22)$$

and

$$T_1(\lambda,\theta) = \operatorname{Im}\left[\int_0^Z g(Z)dZ\right]$$

$$= \frac{1}{\sqrt{2}}\left[-(1 - 2e^\lambda\cos\theta) + (1 - 4e^\lambda\cos\theta + 4e^{2\lambda})^{\frac{1}{2}}\right]^{\frac{1}{2}} - \frac{\theta}{2} - \left(\frac{2+\pi}{2}\right) + \frac{1}{2}\tan^{-1}$$

$$\times\left\{\frac{2^{\frac{1}{2}}e^\lambda\sin\theta + \left[-(1 - 2e^\lambda\cos\theta)(1 - 4e^\lambda\cos\theta + 4e^{2\lambda})^{\frac{1}{2}}\right]^{\frac{1}{2}}}{2^{\frac{1}{2}}(e^\lambda\cos\theta - 1) - \left[(1 - 2e^\lambda\cos\theta) + (1 - 4e^\lambda\cos\theta + 4e^{2\lambda})^{\frac{1}{2}}\right]^{\frac{1}{2}}}\right\} \qquad (23)$$

The values of $T_0(\lambda,\theta)$, (10), and of $T_1(\lambda,\theta)$ for various values of (λ,θ) are given in table II.

The functions g, $T_0(\lambda,\theta)$, and $T_1(\lambda,\theta)$ can be represented by the following series. For $-0.691 < \lambda < 0$:

$$
\left.
\begin{aligned}
g &= i\sqrt{2}\left[e^{Z/2} - \frac{3}{4}e^{-Z/2} - \frac{5}{32}e^{-3Z/2} - \frac{7}{128}e^{5Z/2} \right. \\
&\qquad \left. - \frac{45}{2048}e^{-7Z/2} - \ldots\right] \\
T_o &= \sqrt{2}\left[e^{\lambda/2}\cos\left(\frac{\theta}{2}\right) - \frac{3}{4}e^{-\lambda/2}\cos\left(\frac{\theta}{2}\right) - \frac{5}{32}e^{-3\lambda/2}\cos\left(\frac{3\theta}{2}\right)\right. \\
&\qquad \left. - \frac{7}{128}e^{-5\lambda/2}\cos\left(\frac{5\theta}{2}\right) - \frac{45}{2048}e^{-7\lambda/2}\cos\left(\frac{7\theta}{2}\right) + \ldots\right] \\
T_1 &= \sqrt{2}\left[2\left(e^{\lambda/2}\cos\left(\frac{\theta}{2}\right) - 1\right) + \frac{3}{4}\times\frac{2}{1}\left(e^{-\lambda/2}\cos\left(\frac{\theta}{2}\right) - 1\right)\right. \\
&\qquad + \frac{5}{32}\times\frac{2}{3}\left(e^{-3\lambda/2}\cos\left(\frac{3\theta}{2}\right) - 1\right) + \frac{7}{128}\times\frac{2}{5}\left(e^{-5\lambda/2}\cos\left(\frac{5\theta}{2}\right) - 1\right) \\
&\qquad \left. + \frac{45}{2048}\times\frac{2}{7}\left(e^{-7\lambda/2}\cos\left(\frac{7\theta}{2}\right) - 1\right) + \ldots\right]
\end{aligned}
\right\} \quad (24)
$$

For $\lambda < -0.691$:

$$
\left.
\begin{aligned}
g &= -2\left[1 + \frac{1}{2}e^{2Z} + e^{3Z} + \frac{15}{8}e^{4Z} + \frac{7}{2}e^{5Z} + \ldots\right] \\
T_o &= -2\left[\frac{1}{2}e^{2\lambda}\sin 2\theta + e^{3\lambda}\sin 3\theta + \frac{15}{8}e^{4\lambda}\sin 4\theta\right. \\
&\qquad \left. + \frac{7}{2}e^{5\lambda}\sin 5\theta + \ldots\right] \\
T_1 &= -2\left[\theta + \frac{1}{4}e^{2\lambda}\sin(2\theta) + \frac{1}{3}e^{3\lambda}\sin(3\theta) + \frac{15}{32}e^{4\lambda}\sin(4\theta)\right. \\
&\qquad \left. + \frac{7}{10}e^{5\lambda}\sin(5\theta) + \ldots\right] + 1.14
\end{aligned}
\right\} \quad (25)
$$

Step II.- The construction of the flow in the logarithmic plane can be conveniently performed on specially scaled graph paper. One (Cartesian) coordinate axis is taken as the θ-axis, the other as the M-axis. In addition to the values of M scales showing the corresponding values of $\lambda(M)$ and v/a_o should be indicated. Such scales may be prepared once and for all, for each given value of k. (See (11) and (13).) In diagrams Ia and Ib these scales are drawn for $k = -0.5$ and $k = 1.4$.

The values found for $\operatorname{Im} g^{(n)}$ are entered on this paper and the lines

$$\operatorname{Im} g^{(n)}(\lambda - i\theta) = \text{constant} \tag{26}$$

are drawn. In this way there is obtained by graphical interpolation the values of the functions

$$T_n(M,\theta) = \operatorname{Im}\left\{g^{(n)}\left[\lambda(M) - i\theta\right]\right\} \tag{27}$$

The function $\psi(M,\theta)$ is determined by formula (17). In applying this formula it is necessary to choose a definite value for the arbitrary constant α. In general, it is possible to choose α so that the terms in the series (17) become small for large value of M. In the example under consideration α is set equal to 0.1.

In order to obtain the function $\psi(M,\theta)$ it is necessary to evaluate the products

$$\frac{1}{2}\, Q^{(1)}(M)T_1(M,\theta),\ \frac{3}{4}\, Q^{(2)}(M)T_2(M,\theta),\ . . . \tag{28}$$

The values of the functions $Q^{(n)}(M)$ can be tabulated once and for all (for a given value of k). The above products can be evaluated graphically, by means of a simple nomogram, shown in figure 4.

Adding a finite number of the terms in the series (16) and multiplying by $H(M)$ (a function tabulated once and for all) (see tables Ia, Ib) gives an approximate expression for $\psi(M,\theta)$. Finally, the lines $\psi(M,\theta) = \text{constant}$ (stream lines) are drawn.

For function (8) $\psi(M,\theta)$ has been computed for $k = -0.5$ and $k = 1.4$. For $k = -0.5$ the series has been replaced by

$$\psi_1(M,\theta) \equiv \psi(M,\theta;\ -0.5)$$
$$= \mathrm{Im}\left\{H(2\lambda - 0.2)\left[T_0(M,\theta) + \frac{1}{2}\,Q^{(1)}(2\lambda - 0.2)T_1(M,\theta)\right]\right\},\quad \lambda = \lambda(M) \tag{29}$$

For $k = 1.4$ the series has been replaced by

$$\psi_2(M,\theta) = \psi(M,\theta;\ 1.4)$$
$$= \mathrm{Im}\left\{H(2\lambda - 0.2)\left[T_0(M,\theta) + \frac{1}{2}\,Q^{(1)}(2\lambda - 0.2)T_1(M,\theta) + \frac{3}{4}\,Q^{(2)}(2\lambda - 0.2)T_2(M,\theta)\right]\right\},\quad \lambda = \lambda(M) \tag{30}$$

The streamlines (in the logarithmic plane) are shown in figure 6 $(k = -0.5)$ and figure 7 $(k = 1.4)$.

Step III.- The flow is first transferred from the logarithmic plane to the hodograph plane. This is done by using (for the hodograph plane) polar coordinate graph paper. The polar coordinates in the hodograph plane are v/a_0 and θ. Since for each plane the values of v/a_0 and θ are known, the transition presents no difficulties.

The hodograph streamlines of the flow generated by the function (8) are shown in figure 8 (for $k = -0.5$) and in figure 9 (for $k = 1.4$).

In order to obtain the streamlines in the physical plane it is necessary to compute the integrals (19) along the streamlines in the hodograph plane.

The partial derivatives of ψ entering in the integrands can be expresses as follows

$$\psi_v = R^{(0)}\,\mathrm{Im}\,g_Z + R^{(1)}\,\mathrm{Im}\,g + \frac{1}{2}\,R^{(2)}\,\mathrm{Im}\,g^{(1)} + \frac{3}{4}\,R^{(3)}\,\mathrm{Im}\,g^{(2)} + \ldots \tag{31}$$

(See reference 1, sec. 12.)

where

$$\left.\begin{aligned} R^{(o)} &= H\frac{d\lambda}{dv}, \quad R^{(1)} = H_v + \frac{1}{2}HQ^{(1)}\frac{d\lambda}{dv}, \quad \ldots \\ R^{(n)} &= \left[\frac{(2n-2)!}{2^{2n-2}(n-1)!}\left(HQ^{(n-1)}\right)_\lambda + \frac{2n(2n-1)}{4n}HQ^{(n)}\right]\frac{d\lambda}{dv} \\ & \qquad n = 2,3, \ldots \end{aligned}\right\}(32)$$

and

$$\psi_\theta = H\left[\operatorname{Re} g_z + \frac{1}{2}Q^{(1)}\operatorname{Re} g + \frac{3}{4}Q^{(2)}\operatorname{Re} g^{(1)} + \frac{15}{8}Q^{(3)}\operatorname{Re} g^{(2)} + \ldots\right] \quad (33)$$

The values of $R^{(o)}, \ldots, R^{(4)}$ for $k = 1.4$ are given in table Ib.

The formulas for the derivatives of T_0, T_I for the case of the function (8) are given in appendix II. The values of the derivatives of ψ are given in table III.

The function ρ which enters in the integrand also can be tabulated once and for all.

The integration is to be performed graphically. When a number of streamlines in the physical plane are drawn, a rather complete picture of the flow is obtained. In fact, for each point in the physical plane for which the corresponding point in the hodograph plane is known, the value of the speed, pressure, local Mach number, and so forth, also is known.

For the function (8) it turns out that a part of the streamlines $\psi = 0$ forms a closed contour. The streamline starts at $-\infty$ and divides itself into two branches at the first stagnation point. The two parts come together at the second stagnation point. Thus a flow around an obstacle has been obtained.[1] The boundary of the obstacle (for $k = -0.5$) is shown in figure 10.

[1]For $k = 1.4$, see p. 21.

4. Determination of the Function $g(Z)$

Which Leads to a Flow of Desired Type

It has been shown that every analytic function $g(Z)$ leads to a compressible flow in the z-plane (physical plane). The type of the flow obtained thus depends upon the choice of $g(Z)$.

If it is desired to obtain a flow around a given obstacle, or at least around an obstacle similar to a given one, it is well to proceed as follows:

First, determine the incompressible flow around the given obstacle and obtain the complex potential as a function of $\log v - i\theta$. (By using Theodorsen's method (reference 3) this problem can always be solved.) This function, $g(Z)$, is transformed into a stream function of a compressible flow. Assume that the compressible flow in the z-plane is a flow around a closed body.[1]

Let B be the original profile, B_1 the boundary of the obstacle of the compressible flow. Clearly, B_1 will be different from B. For instance, if B is a circle, there is obtained (for certain values of the arbitrary constant α entering in the computations) the profile shown in figure 10. The profile distortion can be represented by associating to each point z of B a point z of B_1 and writing

$$z_1 = z + d$$

(z, z_1, d are complex numbers). Now, the profile B^* the points of which are given by

$$z^* = z - d$$

can be described as being distorted "in the opposite direction." If one were to construct the incompressible flow

[1]Necessary and sufficient conditions for a function $\psi(v/a_0, \theta)$ to lead to a flow around a closed body are derived in reference 1, sec. 14. If the original profile is symmetric with respect to both the real and the imaginary axis, the compressible flow pattern will always flow around a closed profile.

around B^*, compute its complex potential in the logarithmic plane, $g^*(Z)$, and use this function to obtain a compressible flow, it might be expected that a flow around a profile B_2 would be obtained which better approximates B. This procedure may be repeated.

In general, the function $g(Z)$ will be of a very complicated form, even if the complex potential in the physical plane is a simple expression. Therefore, it is convenient to know functions which can be expressed in a closed form by means of elementary functions, and which are potentials of incompressible flows around closed profiles (in the physical plane). Each such function leads to compressible flows of similar type, and the computation of these flows is relatively simple.

A function of the desired type depending upon three arbitrary parameters will presently be constructed. By the following successive transformations, g as a function of Z is determined:

$$g = iZ_1 \tag{34}$$

$$Z_2 = \frac{AZ_1}{C + Z_1}, \quad 0 < A < \infty, \quad 0 < C < \infty \tag{35}$$

$$Z_3 = \sqrt{Z_2^2 + \sqrt{Z_2^4 + 2Z_2^2}} \tag{36}$$

$$Z_4 = -i \log \frac{m + iZ_3}{m - iZ_3}, \quad m \text{ real} \tag{37}$$

$$Z_5 = \left(Z_4 - \frac{\pi}{2}\right)^2 \tag{38}$$

$$Z = \log Z_5 \tag{39}$$

Here, A, C, m are arbitrary constants. The right half plane of the Z_1-plane is taken into the domains indicated in figures 11b to 11d, and into the domain indicated in figure 12. Now, consider a Riemann surface consisting of two sheets spread over the domain of the Z_5-plane (fig. 12) and possessing a branch point at the point α (corresponding to $g = \infty$). Function $g(Z_5)$ maps this Riemann surface into the exterior of a slit along the real axis (in the g-plane). Therefore the domain in Z_5-plane can be interpreted as a hodograph of a circulation-free flow around a slender symmetric body with a sharp trailing edge, and g is the complex potential of the flow (provided the constants are chosen in an appropriate way).

The Z-plane is the logarithmic plane for the abovementioned flow and therefore $g(Z)$ can be transformed into a stream function of a compressible flow by the method given in the preceding sections.

The computation will be facilitated by the relatively simple form of g.

CONCLUDING REMARKS

This paper employs the hodograph method introduced into the theory of compressible fluids by Chaplygin (reference 4) in 1902.

An essential feature of the hodograph method is that the principle of superposition of solutions (each of which represents a stream function in the hodograph plane) holds. If $\psi_1(v,\theta)$, $\psi_2(v,\theta)$, . . . are solutions of the differential equation (17) and if A_1, A_2, . . . are arbitrary constants, then $\psi(v,\theta) = \Sigma A_n\psi_n(v,\theta)$ is also a solution of (17). One of the main problems of the theory consists in determining functions $\psi(v,\theta)$ which yield in physical plane flows around closed bodies all streamlines starting and ending at infinity.

Until the present time, no solutions of this kind have been given except that of Von Kármán (reference 5) and Tsien (reference 6), who replaced the correct pressure density relation $p = \sigma\rho^{1.4}$ by $p = A + \sigma/\rho$. This simplification yields a rough approximation and can be employed only when the maximum speed is sufficiently small.

In the present paper a solution which fulfills the conditions described above and which is based on the true pressure density relation is derived for the first time.

Two examples have been considered: one corresponding to a nearly circular domain, the other corresponding to a slender obstacle. In the first case the computation of the stream pattern has been carried out completely.

The foregoing method may be applied to other profiles by either of two alternative schemes. Another analytic function $g(Z)$ may be chosen and the procedure employed in the present paper repeated. A combination of simple solutions, such as those given by Chaplygin (reference 4), Bers and Gelbart (reference 7), or the author (reference 8, p. 23 and reference 9, p. 277) may be added to either of the stream functions found in the afore-mentioned examples. In each instance stream functions may be found that fulfill the previously described conditions.

The method can be extended to partly supersonic flows. One of the advantages of the foregoing method is that a large part of the computation is independent of the function g. These computations can be carried out once and for all and tabulated, which will greatly facilitate the application of the method.

As has been seen in the example under consideration, the actual calculation of the (subsonic) flow corresponding to a given function $g(Z)$ does not entail any theoretical difficulties. On the other hand, it does involve a very considerable amount of numerical computation. It is, consequently, expedient to use special computational devices, such as punch card machines, the differential analyzer, and so forth, in order to overcome these difficulties. The method described in the present section has, however, been developed with the assumption that no such special devices (except perhaps an ordinary computing (multiplication and addition) machines) are available.

In the sequel to this report the author will discuss the use to which these devices may be put in computing compressible fluid flows.[1]

Brown University,
Providence, R. I., May 15, 1944.

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[1]The flow in the physical plane for $k = 1.4$ will be described on occasion as well as several aerodynamic consequences which follow from the present results.

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APPENDIX I

Other Form of Formula (17)

It is often more convenient to use an operator which transforms analytic functions into stream functions and which differs slightly from (17). This operator has the form

$$\psi = \operatorname{Im}\left\{H\left[\int_{-1}^{+1} f\left(\frac{1}{2}Z(1-t^2)\right)dt/\left(1-t^2\right)^{\frac{1}{2}}\right.\right.$$

$$+ ZQ^{(1)}\int_{-1}^{+1} t^2 f'\left(\frac{1}{2}Z(1-t^2)\right)dt/(1-t^2)^{\frac{1}{2}}$$

$$\left.\left. + Z^2Q^{(2)}\int_{-1}^{+1} t^4 f\left(\frac{1}{2}Z(1-t^2)\right)dt/(1-t^2)^{\frac{1}{2}} + \ldots\right]\right\}$$

This operator can be obtained from (17) by setting

$$g(Z) = \int_{-1}^{+1} f\left(\frac{1}{2}Z(1-t^2)\right)dt/(1-t^2)^{\frac{1}{2}}$$

or

$$f(Z) = \frac{2}{\pi}\int_{0}^{\pi/2} Z\sin\vartheta\,\frac{dg(2Z\sin^2\vartheta)}{d(Z\sin^2\vartheta)}\,d\vartheta + \frac{g(0)}{\pi}$$

APPENDIX II

Formulas for the Derivatives of T for the Function (8)

$$\frac{\partial T_0}{\partial \theta} = -\frac{1}{2}\left[\left(\frac{1}{2} - e^{\lambda} \cos\theta\right) + \left(\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\times \left[1 - \frac{1}{2}\left(\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right)\right]^{-\frac{1}{2}} + \frac{e^{\lambda}\sin\theta}{4}\left[-\left(\frac{1}{2} - e^{\lambda} \cos\theta\right)\right.$$

$$\left. + \left(\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \left[\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right]^{-3/2}$$

$$\frac{\partial T_0}{\partial \lambda} = -\cot\theta \frac{\partial T_0}{\partial \theta} + \frac{e^{2\lambda}}{2}\left[\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right]^{-\frac{1}{2}}$$

$$\times \left[1 - \frac{1}{2}\left(\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right)^{-\frac{1}{2}}\right]$$

$$\times \left[\frac{1}{\left[-\left(\frac{1}{2} - e^{\lambda} \cos\theta\right) + \left(\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}\right]$$

$$+ \frac{e^{2\lambda}}{2} \frac{\left[-\left(\frac{1}{2} - e^{\lambda} \cos\theta\right) + \left(\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{\left(\frac{1}{4} + e^{\lambda} \cos\theta + e^{2\lambda}\right)^{3/2}}$$

$$\frac{\partial T_1}{\partial \theta} = -\left[-\left(\frac{1}{2} - e^{\lambda} \cos\theta\right) + \left(\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\times \left[1 + \frac{1}{2}\left(\frac{1}{4} - e^{\lambda} \cos\theta + e^{2\lambda}\right)^{-\frac{1}{2}}\right]$$

$$\frac{\partial T_1}{\partial \lambda} = T_0$$

Table Ia

($k = -0.5$; $\alpha = 0.1$)

Functions used in computing the stream functions of a compressible flow.

$2\lambda-2\alpha$	M	v/a_o	H	$Q^{(1)}$
-2.82	0.265	0.272	1.000	-0.0299
-1.96	0.403	0.430	1.000	-0.0602
-1.612	0.473	0.518	1.002	-0.0909
-1.51	0.497	0.551	1.003	-0.1115
-1.388	0.528	0.593	1.004	-0.1267
-0.79	0.690	0.861	1.015	-0.3009
-0.394	0.820	1.165	1.046	-0.7033
-0.188	0.901	1.440	1.111	-1.5871
-0.132	0.921	1.527	1.145	-2.2682

Table Ib. - The values of F, H, $Q^{(n)}$, $R^{(n)}$ (for $k = 1.4$, $\alpha = 0.1$): Functions used in computing the stream-function of a compressible flow.

$2\lambda-2\alpha$	M	v/a_o	F	H	$Q^{(1)}$	$Q^{(2)}$	$Q^{(3)}$	$Q^{(4)}$	$R^{(o)}$	$R^{(1)}$	$R^{(2)}$	$R^{(3)}$
$-\infty$	.0000	.0000	.0000	1.0000	.0000	.0000	.0000	.0000		.0000	.0000	.0000
-3.8772	.1000	.0999	.0001	1.0000	.0000	.0001	-.0001	.0002	9.9600	.0010	-.0020	.0000
-2.5096	.2000	.1992	.0011	1.0002	-.0009	.0015	-.0013	.0008	4.9197	.0030	-.0103	.0148
-1.7327	.3000	.2972	.0064	1.0014	-.0056	.0085	-.0083	.0064	3.2141	.0090	-.0411	.0670
-1.2071	.4000	.3938	.0256	1.0042	-.0199	.0342	-.0395	.0374	2.3371	.0235	-.1203	.2307
-.8238	.5000	.4879	.0866	1.0110	-.0574	.1160	-.1680	.2116	1.7947	.0520	-.3152	.7600
-.6706	.5500	.5341	.1565	1.0167	-.0935	.2101	-.3496	.5211	1.5901	.0755	-.5082	1.4060
-.5364	.6000	.5795	.2839	1.0247	-.1501	.3823	-.7494	1.3527	1.4150	.1088	-.8277	2.6949
-.4204	.6500	.6242	.5245	1.0359	-.2401	.7089	-1.6847	3.7802	1.2620	.1566	-1.3796	5.4353
-.3203	.7000	.6680	1.0060	1.0515	-.3870	1.3663	-4.0770	11.7558	1.1254	.2275	-2.3946	11.8168
-.2207	.7500	.7110	2.0623	1.0811	-.6959	2.8304	-12.7121	51.4676				
-.1615	.8000	.7532	4.6583	1.1049	-1.0896	6.4088	-35.1357	194.4562	.8847	.5248	-9.0769	81.6373
-.1015	.8500	.7945	12.5662	1.1517	-2.0186	17.4338	-149.6938	954.643	.7717	.8743	-21.9464	306.6745
-.0535	.9000	.8349	46.6378	1.2275	-4.3787	65.3783	-997.3960	16412.652	.6555	1.6804	-71.6427	1772.3621
.0000	1.0000	.9129	∞	∞	$-\infty$	∞	$-\infty$	∞	.0000	∞	$-\infty$	∞

TABLE II

Computation of the stream-function (in the logarithmic plane) of a compressible flow generated by the analytic function (8).

$-\lambda$	θ	T_0	T_1	T_2	T_3	$\psi(k=-0.5)$	$\psi(k=1.4)$
0.02	0.05	-0.0376	-0.0020	0.0000	+0.0000	-0.0365	-0.0400
0.02	0.20	+0.0002	-0.0402	-0.0008	+0.0001	+0.0250	+0.0105
0.02	0.40	+0.0968	-0.1550	-0.0049	+0.0021	+0.1950	+0.0994
0.02	0.60	+0.2051	-0.3318	-0.0160	+0.0103	+0.4163	+0.0702
0.02	0.80	+0.2975	-0.5588	-0.0413	+0.0326	+0.6518	-0.3030
0.06	0.08	-0.1159	-0.0044	-0.0003	+0.0000	-0.1162	-0.1212
0.06	0.30	-0.0251	-0.0892	-0.0058	+0.0006	+0.0144	-0.0187
0.06	0.34	-0.0019	-0.1148	-0.0074	+0.0008	+0.0496	+0.0084
0.06	0.40	+0.0345	-0.1582	-0.0105	+0.0016	+0.1062	+0.0453
0.06	0.50	+0.0964	-0.2418	-0.0174	+0.0041	+0.2069	+0.1008
0.06	0.70	+0.2096	-0.4494	-0.0390	+0.0167	+0.4155	+0.1387
0.06	0.90	+0.2973	-0.7016	-0.0755	+0.0466	+0.6181	+0.0038
0.06	1.10	+0.3440	-1.0154	-0.1340	+0.1042	+0.8066	-0.4074
0.10	0.15	-0.1791	-0.0154	-0.0019	+0.0000	-0.1870	-0.1852
0.10	0.35	-0.0625	-0.1200	-0.0127	+0.0005	+0.0219	-0.0574
0.10	0.50	+0.0414	-0.2446	-0.0246	+0.0029	+0.1258	+0.0536
0.10	0.60	+0.1073	-0.3442	-0.0363	+0.0078	+0.2266	+0.1130
0.10	0.70	+0.1664	-0.4564	-0.0505	+0.0135	+0.3248	+0.1652
0.10	1.00	+0.2975	-0.8540	-0.1387	+0.0683	+0.5936	+0.1361
0.20	0.22	-0.3504	-0.0184	-0.0071	-0.0003	-0.3497	-0.3505
0.20	0.40	-0.1802	-0.1468	-0.0255	-0.0017	-0.1534	-0.1853
0.20	0.70	-0.0013	-0.5354	-0.0768	+0.0044	+0.1177	+0.0098
0.20	0.80	+0.1259	-0.5972	-0.0931	+0.0105	+0.2599	+0.1310
0.20	1.00	+0.2204	-0.8790	-0.1323	+0.0313	+0.4179	+0.2138
0.20	1.10	+0.2544	-1.0328	-0.1481	+0.0536	+0.4864	+0.2322
0.30	0.30	-0.4688	+0.0306	+0.0162	-0.0050	-0.4689	-0.4692
0.30	0.75	+0.0099	-0.5364	-0.1034	-0.0008	+0.0886	+0.0164
0.30	0.85	+0.0753	-0.6758	-0.2058	+0.0060	+0.1750	+0.0743
0.30	0.95	+0.1287	-0.8222	-0.2398	+0.0250	+0.2504	+0.1235
0.30	1.10	+0.1904	-1.0546	-0.2491	+0.0325	+0.3467	+0.1888
0.30	1.20	+0.2209	-1.2164	-0.3078	+0.0422	+0.4013	+0.2166
0.40	0.35	-0.5617	-0.0258			-0.5616	-0.5577
0.40	0.60	-0.2034	-0.3274			-0.1682	-0.1543
0.40	0.85	+0.0091	-0.6802			+0.0843	+0.1111
0.40	0.93	+0.0568	-0.8016			+0.1457	+0.1770
0.40	1.05	+0.1147	-0.9900			+0.2247	+0.2632
0.40	1.30	+0.1941	-1.4062			+0.3504	
0.60	0.40	-0.6396	+0.0278			-0.6360	-0.6376
0.60	0.45	-0.5273	-0.0422			-0.5245	-0.5241
0.60	0.60	-0.3070	-0.2742			-0.2892	-0.2864
0.60	0.75	-0.1572	-0.5112			-0.1239	-0.1189
0.60	0.95	-0.0208	-0.8364			+0.0336	+0.0419
0.60	1.00	+0.0051	-0.9256			+0.0664	+0.0745
0.60	1.15	+0.0683	-1.1744			+0.1446	
0.80	0.50	-0.3970	-0.0318			-0.3959	-0.3962
0.80	0.70	-0.2158	-0.3878			-0.2022	-0.2148
0.80	0.90	-0.0898	-0.7378			-0.0640	-0.0880
0.80	1.05	-0.0209	-1.0040			+0.0143	-0.0169
0.80	1.10	-0.0017	-1.0934			+0.0366	+0.0010
0.80	1.20	+0.0314	-1.2724			+0.0760	+0.0347
0.80	1.40	+0.0801	-1.6374			+0.1374	
0.80	-0.02	-0.2981	+1.1928			-0.2563	-0.3011
1.00	0.54	-0.2520	-0.0436			-0.2335	-0.2519
1.00	0.75	-0.1561	-0.4426			-0.1452	-0.1550
1.00	1.00	-0.0389	-0.9050			-0.0362	-0.0366
1.00	1.10	-0.0038	-1.0902			-0.0035	-0.0011
1.00	1.20	+0.0015	-1.2752			+0.0257	+0.0047
1.00	1.35	+0.0343	-1.5564			+0.0621	+0.0382
1.00	0.00	-0.3079	+1.4728			-0.2306	-0.3116
1.20	0.58	-0.1515	-0.0828			-0.1502	-0.1513
1.20	0.80	-0.1022	-0.5116			-0.0945	-0.1009
1.20	1.05	-0.0426	-0.9890			-0.0277	-0.0401
1.20	1.15	-0.0214	-1.1792			-0.0037	-0.0184
1.20	1.22	-0.0080	-1.3126			+0.0117	-0.0048
1.20	1.30	+0.0059	-1.4650			+0.0279	+0.0096
1.20	1.40	+0.0212	-1.6566			+0.0462	+0.0253
1.20	-0.04	-0.2140	+1.2460			-0.0401	-0.0245

Table III

($k = -0.5$; $\alpha = +0.1$)

Computation of the profile in the physical plane. The flow is generated by the function (8). The values given below are computed for points where $\psi = 0$ (approximately).

λ	$\lambda - \alpha$	θ	ψ_v	ψ_θ	$dx/d\lambda$	$dy/d\lambda$
-0.194	-0.294	0.70	0.3669	0.8476	-0.5699	-0.4800
-0.060	-0.160	0.40	0.3986	0.8505	-0.4339	-0.1834
-1.000	-1.100	1.20	0.1251	0.3829	-0.3485	-0.8962
-0.060	-0.160	0.30	0.1044	1.1894	-0.5190	-0.1605
-0.094	-0.194	0.50	0.2309	0.9525	-0.4740	-0.2590
-0.600	-0.700	1.00	0.4385	0.5050	-0.6850	-1.0669
-1.410	-1.510	1.30	-0.0165	0.2453	-0.2385	-0.8590
-2.000	-2.100	1.30	-0.0902	0.0272	-0.5305	-1.9110
-1.000	-1.100	-0.03	6.9885	-1.8565	64.3002	-1.9298
-1.700	-1.800	-0.10	-0.0062	-0.1291	0.6075	-0.0609
-2.000	-2.100	-0.15	-0.0792	-0.0572	1.0191	-0.1540
-0.720	-0.820	-0.02	-12.9584	-146.8974	243.4653	-4.8703
-0.094	-0.194	0.36	0.3748	1.0478	-0.5913	-0.2259
-0.720	-0.820	-0.02	-12.9584	-146.8974	243.4653	-4.8703
-0.094	-0.194	0.36	0.3748	1.0478	-0.5913	-0.2259
-0.197	-0.297	0.51	0.5404	0.8531	-0.7775	-0.4400
-0.396	-0.496	0.78	0.5105	0.6564	-0.8152	-0.8065
-1.000	-1.100	1.08	0.1594	0.2950	-0.4009	-0.7593
-1.410	-1.510	1.17	-0.0153	0.1368	-0.1885	-0.4448
0.000	-0.100	0.00		0.6125		

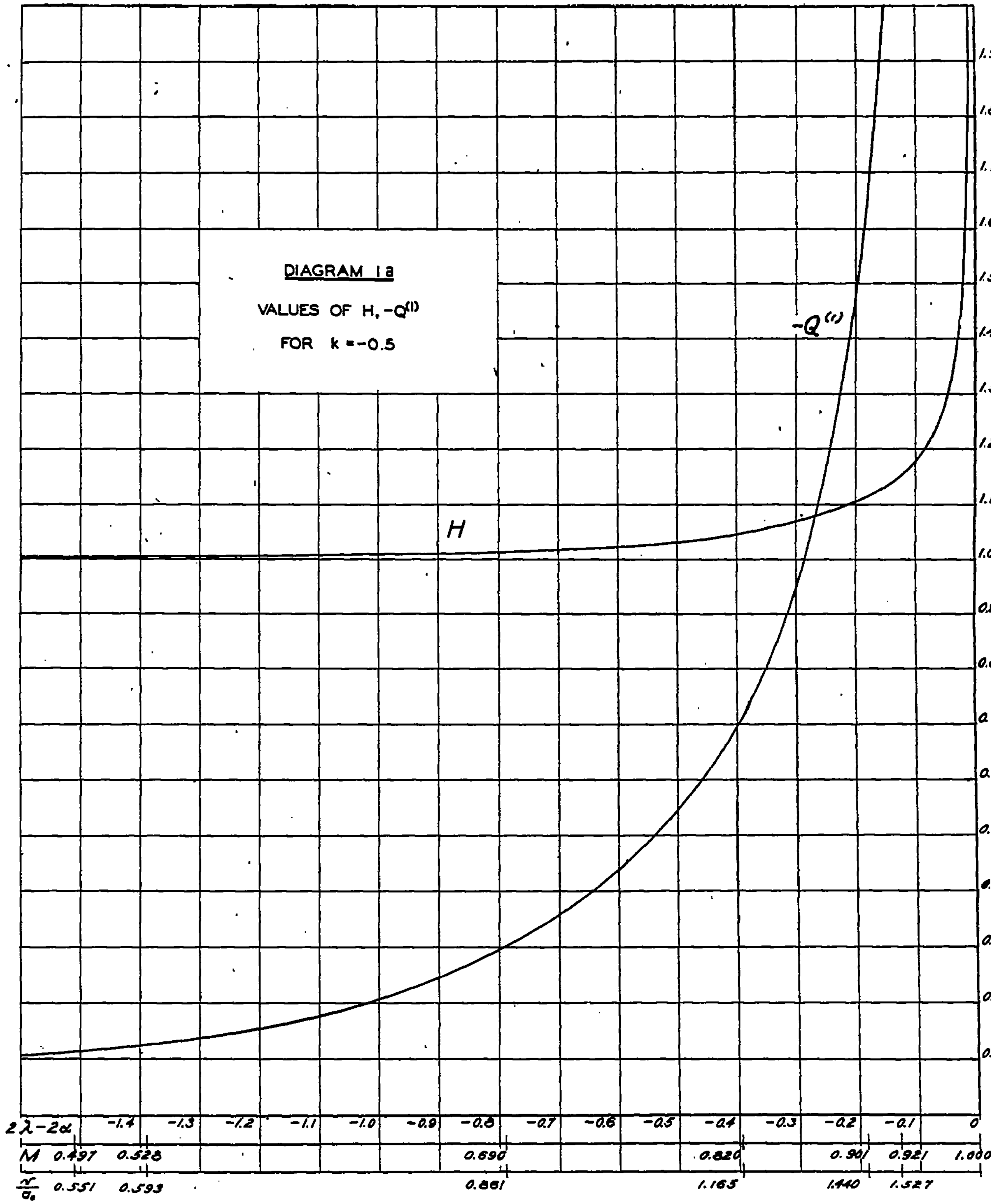
DIAGRAM Ia
VALUES OF H, -Q(1)
FOR k = -0.5
-Q(1)
H
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2λ-2α -1.4 -1.3 -1.2 -1.1 -1.0 -0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0
M 0.497 0.528 0.690 0.820 0.901 0.921 1.000
0.551 0.593 0.861 1.165 1.440 1.527

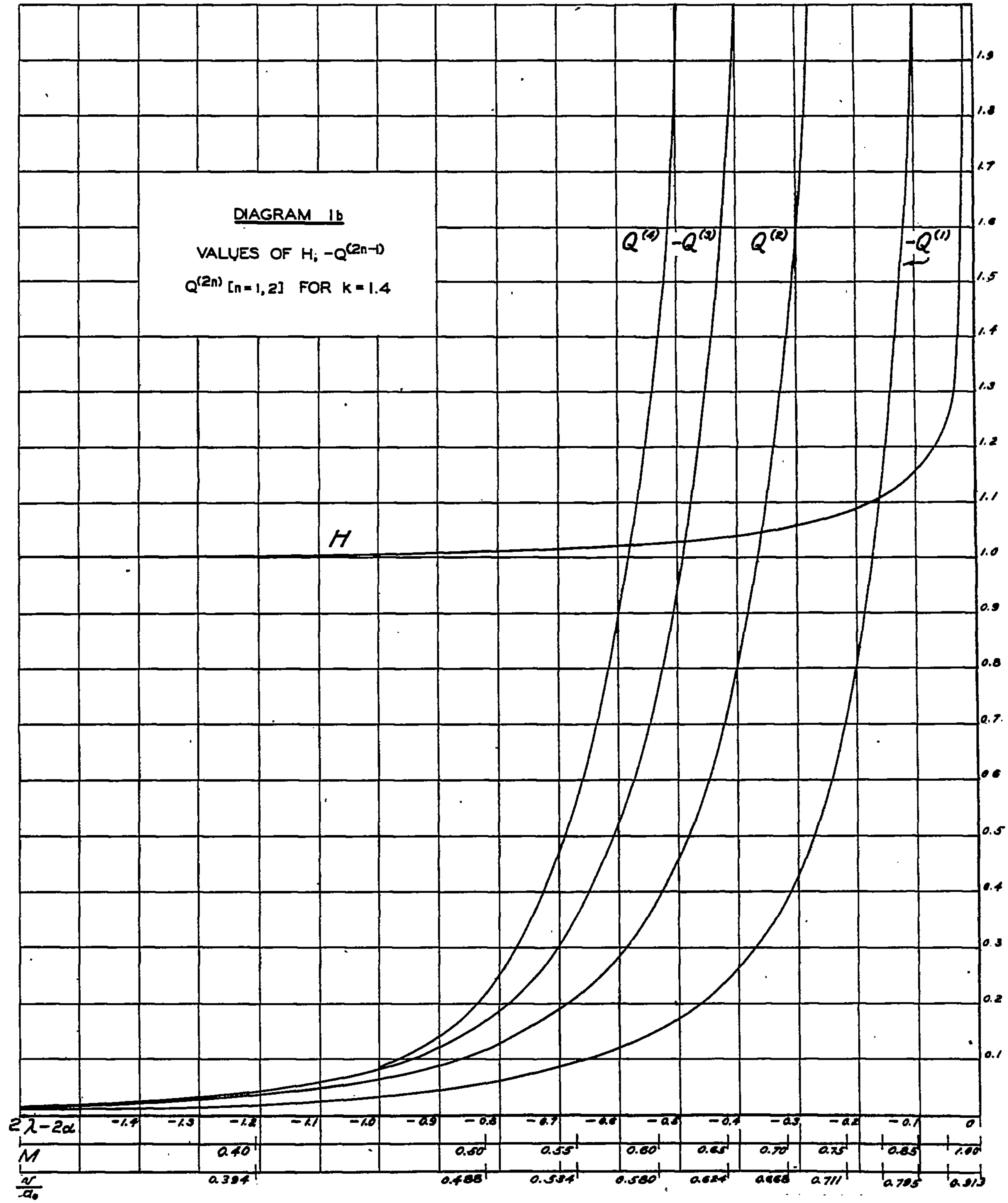
DIAGRAM Ib
VALUES OF H; $-Q^{(2n-1)}$
$Q^{(2n)}$ [n = 1, 2] FOR k = 1.4
H
$Q^{(4)}$
$-Q^{(3)}$
$Q^{(2)}$
$-Q^{(1)}$
$2\lambda - 2\alpha$
M
$\frac{w}{a_0}$

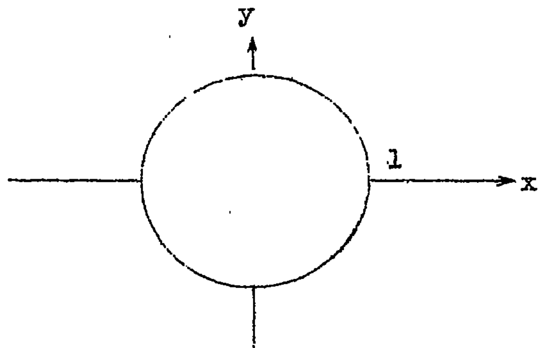

Figure 1

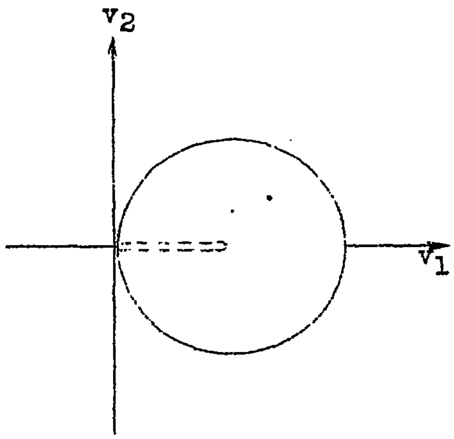

Figure 2

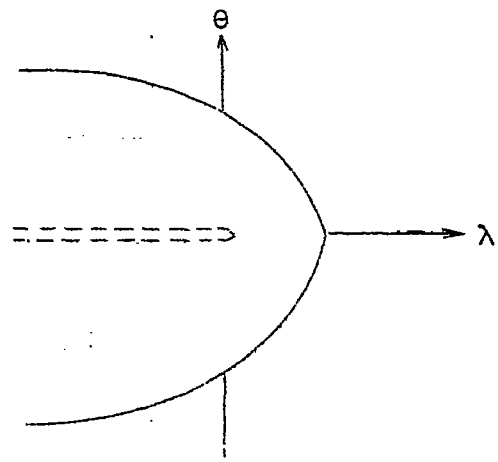

Figure 3

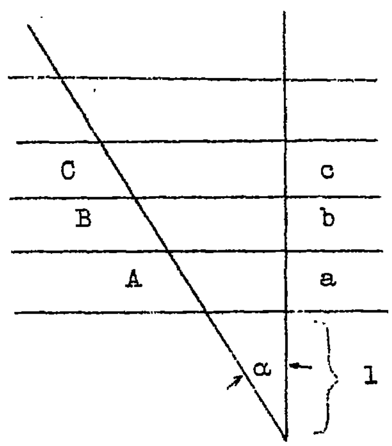

Figure 4.- Nomogram for the products (28). $\cos\alpha = 1/Q(M)$. If $\overline{Oa} = T(M,\theta_1)$, $\overline{Ob} = T(M,\theta_2)$, ..., then $\overline{OA} = Q(M)T(M,\theta_1)$, $\overline{OB} = Q(M)T(M,\theta_2)$, ... An obvious adjustment is made when $Q(M) < 1$. By this procedure we may obtain any of the products $Q^{(v)}T_\nu$.

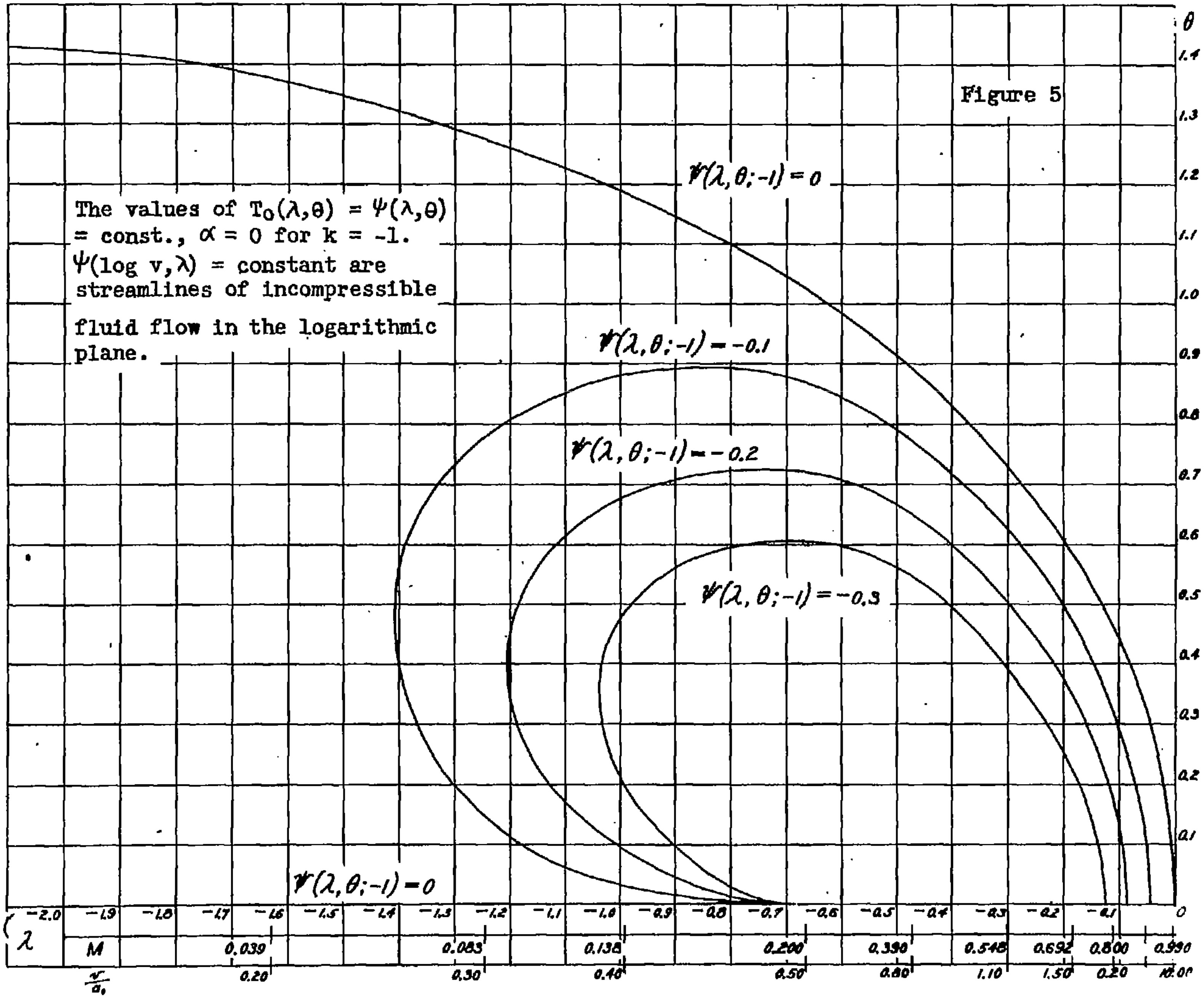
Figure 5
The values of $T_0(\lambda,\theta) = \psi(\lambda,\theta)$ = const., $\alpha = 0$ for k = -1. $\psi(\log v,\lambda)$ = constant are streamlines of incompressible fluid flow in the logarithmic plane.
$\psi(\lambda,\theta;-1) = 0$
$\psi(\lambda,\theta;-1) = -0.1$
$\psi(\lambda,\theta;-1) = -0.2$
$\psi(\lambda,\theta;-1) = -0.3$
$\psi(\lambda,\theta;-1) = 0$
θ
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
λ
-2.0 -1.9 -1.8 -1.7 -1.6 -1.5 -1.4 -1.3 -1.2 -1.1 -1.0 -0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0
M
0.039 0.083 0.138 0.200 0.390 0.548 0.692 0.800 0.990
$\frac{v}{a_0}$
0.20 0.30 0.40 0.50 0.80 1.10 1.50 0.20 10.00

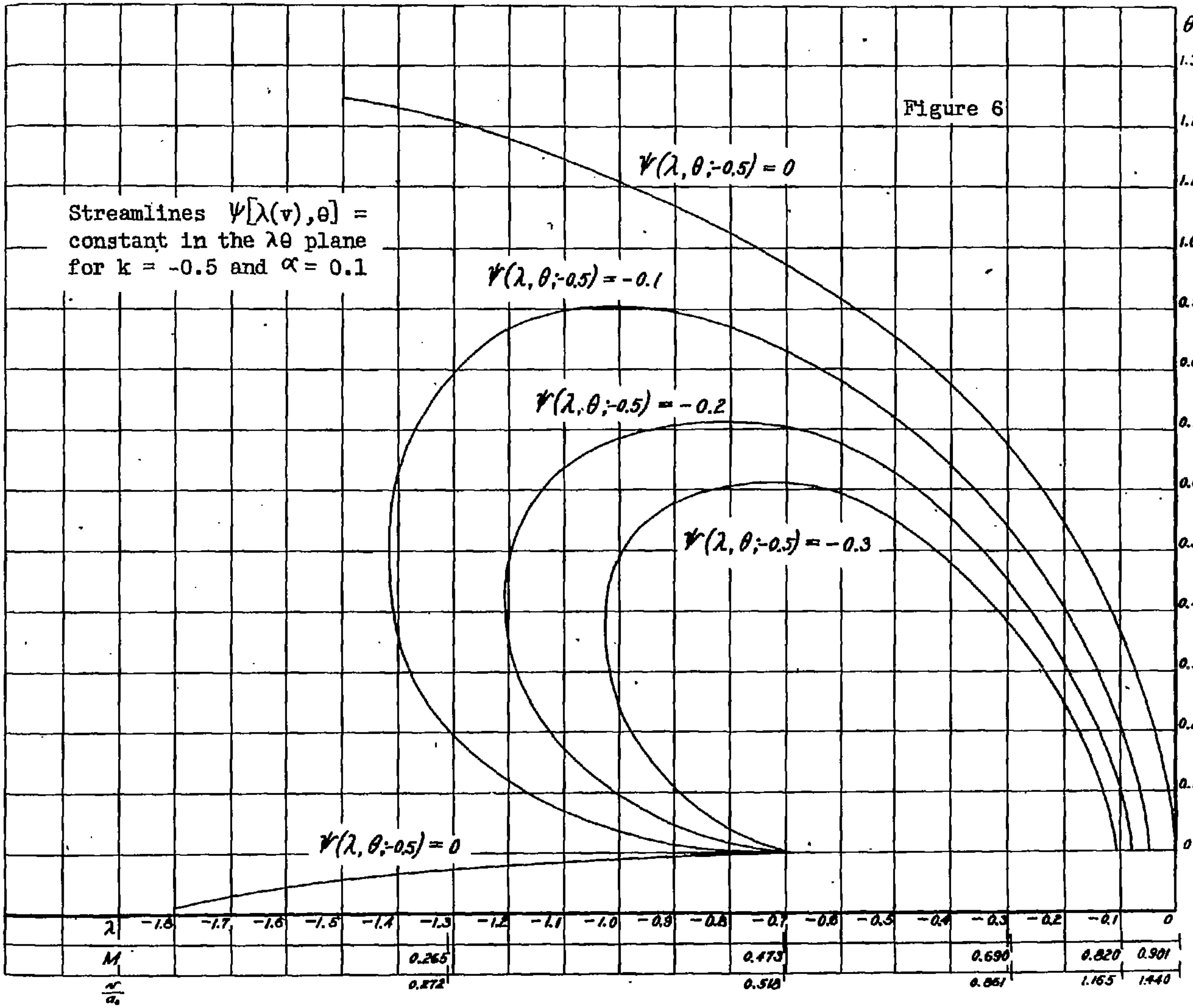
Figure 6
Streamlines $\psi[\lambda(v),\theta]$ = constant in the $\lambda\theta$ plane for k = -0.5 and α = 0.1
$\psi(\lambda,\theta;-0.5) = 0$
$\psi(\lambda,\theta;-0.5) = -0.1$
$\psi(\lambda,\theta;-0.5) = -0.2$
$\psi(\lambda,\theta;-0.5) = -0.3$
$\psi(\lambda,\theta;-0.5) = 0$
θ
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
λ -1.8 -1.7 -1.6 -1.5 -1.4 -1.3 -1.2 -1.1 -1.0 -0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0
M 0.265 0.473 0.690 0.820 0.901
$\frac{v}{a_0}$ 0.272 0.518 0.861 1.165 1.440

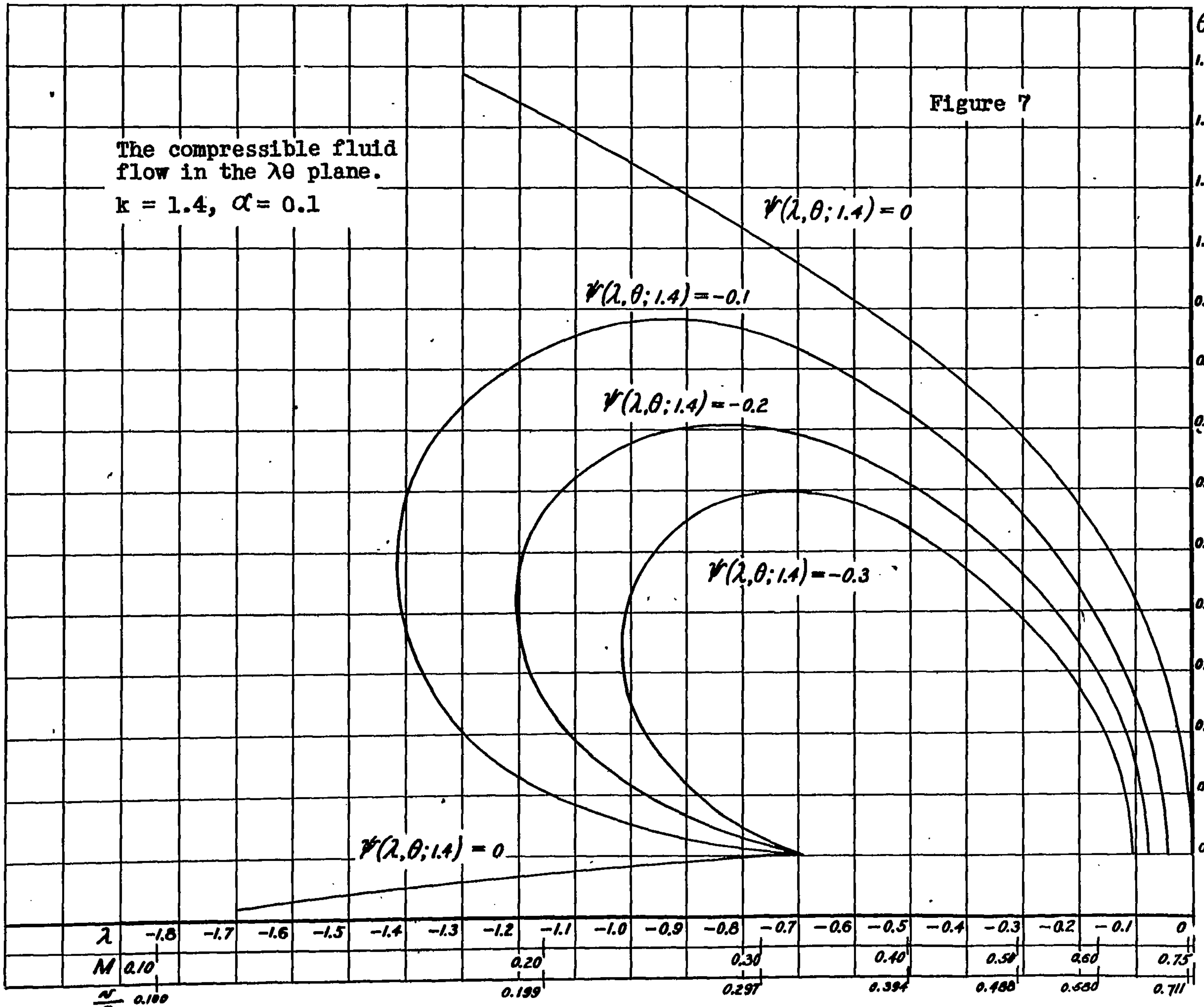
Figure 7
The compressible fluid flow in the λθ plane.
k = 1.4, α = 0.1
ψ(λ,θ;1.4) = 0
ψ(λ,θ;1.4) = -0.1
ψ(λ,θ;1.4) = -0.2
ψ(λ,θ;1.4) = -0.3
ψ(λ,θ;1.4) = 0
θ
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
λ -1.8 -1.7 -1.6 -1.5 -1.4 -1.3 -1.2 -1.1 -1.0 -0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0
M 0.10 0.20 0.30 0.40 0.50 0.60 0.75
v/a₀ 0.100 0.199 0.297 0.394 0.488 0.580 0.711

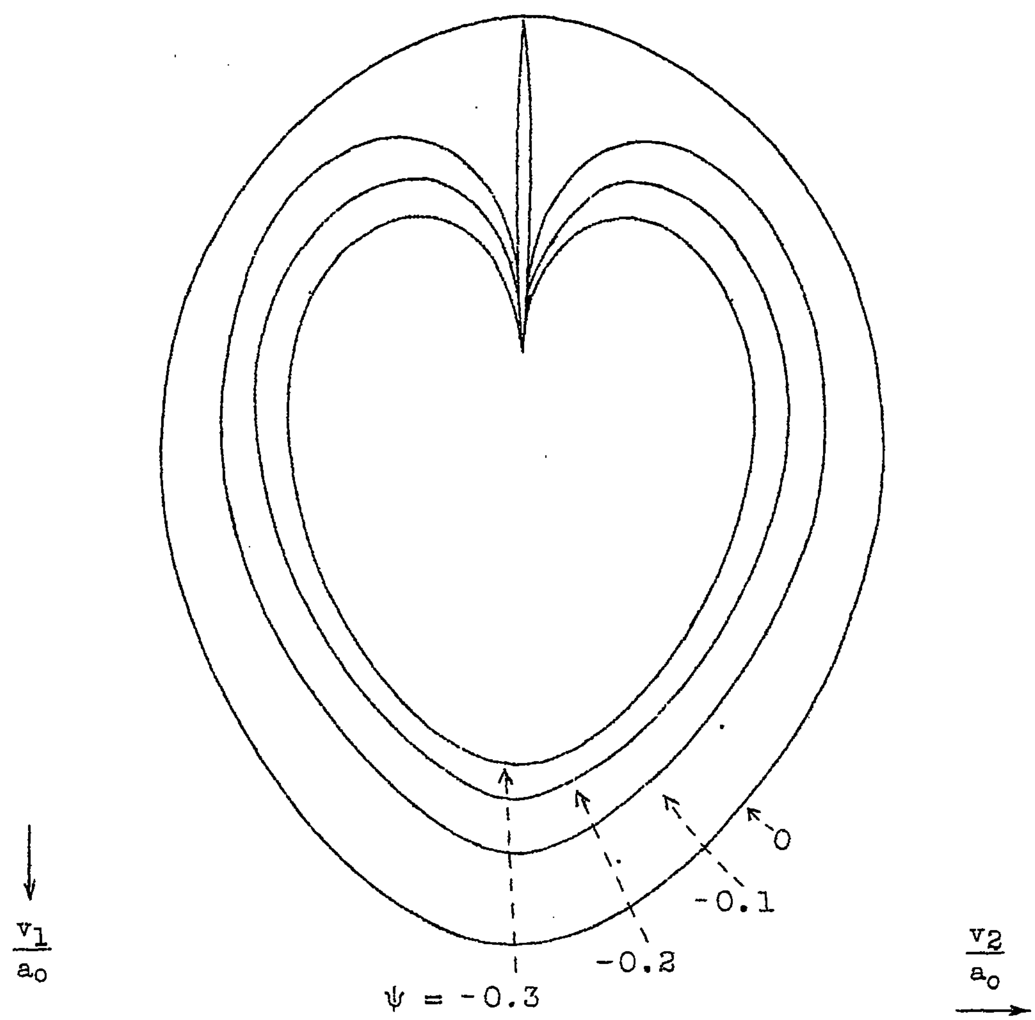

Figure 8.- The hodograph of the compressible fluid flow, $k = -0.5$, $\alpha = 0.1$, $M_{max} = 0.9$, $M_{\infty} = 0.52$.

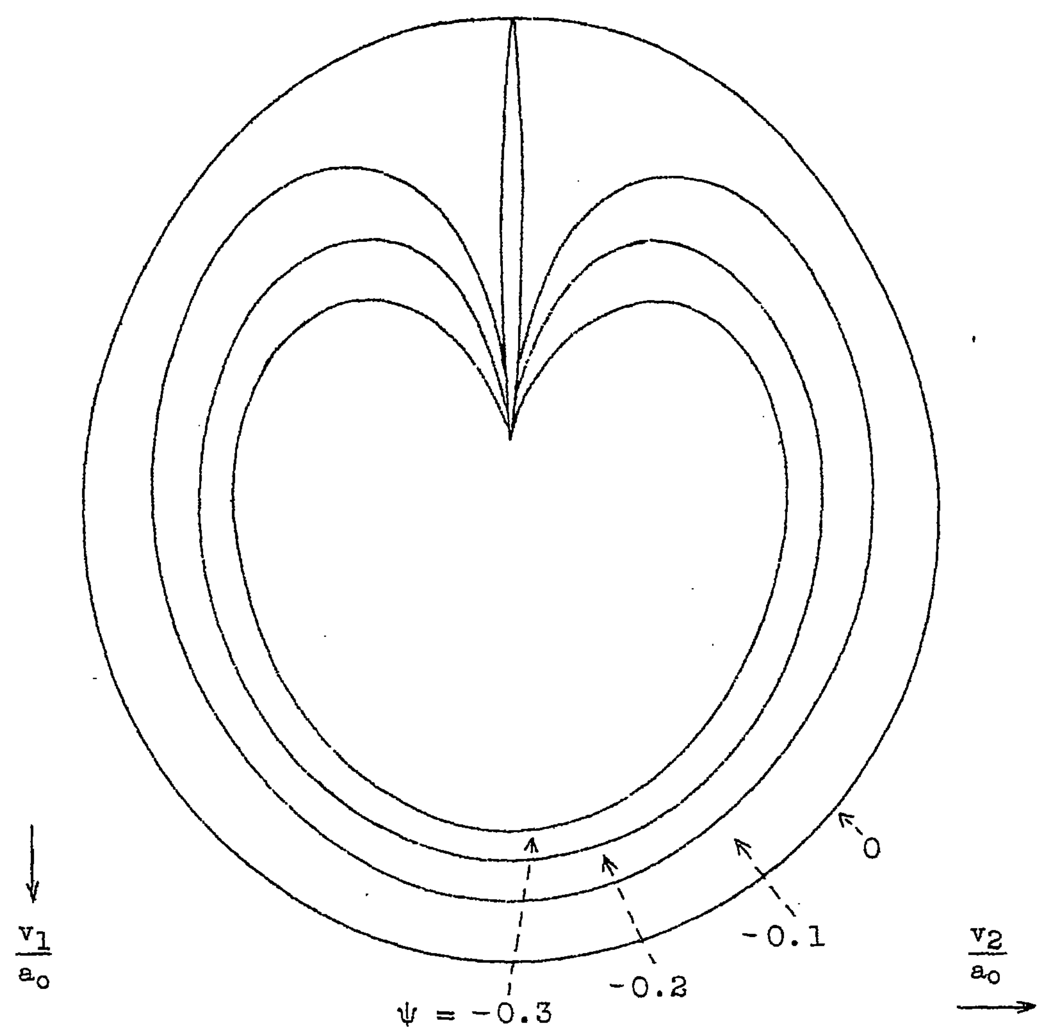

Figure 9.- The hodograph of the compressible fluid flow. $k = 1.4$, $\alpha = 0.1$, $M_{max} = 0.77$, $M_{\infty} = 0.32$.

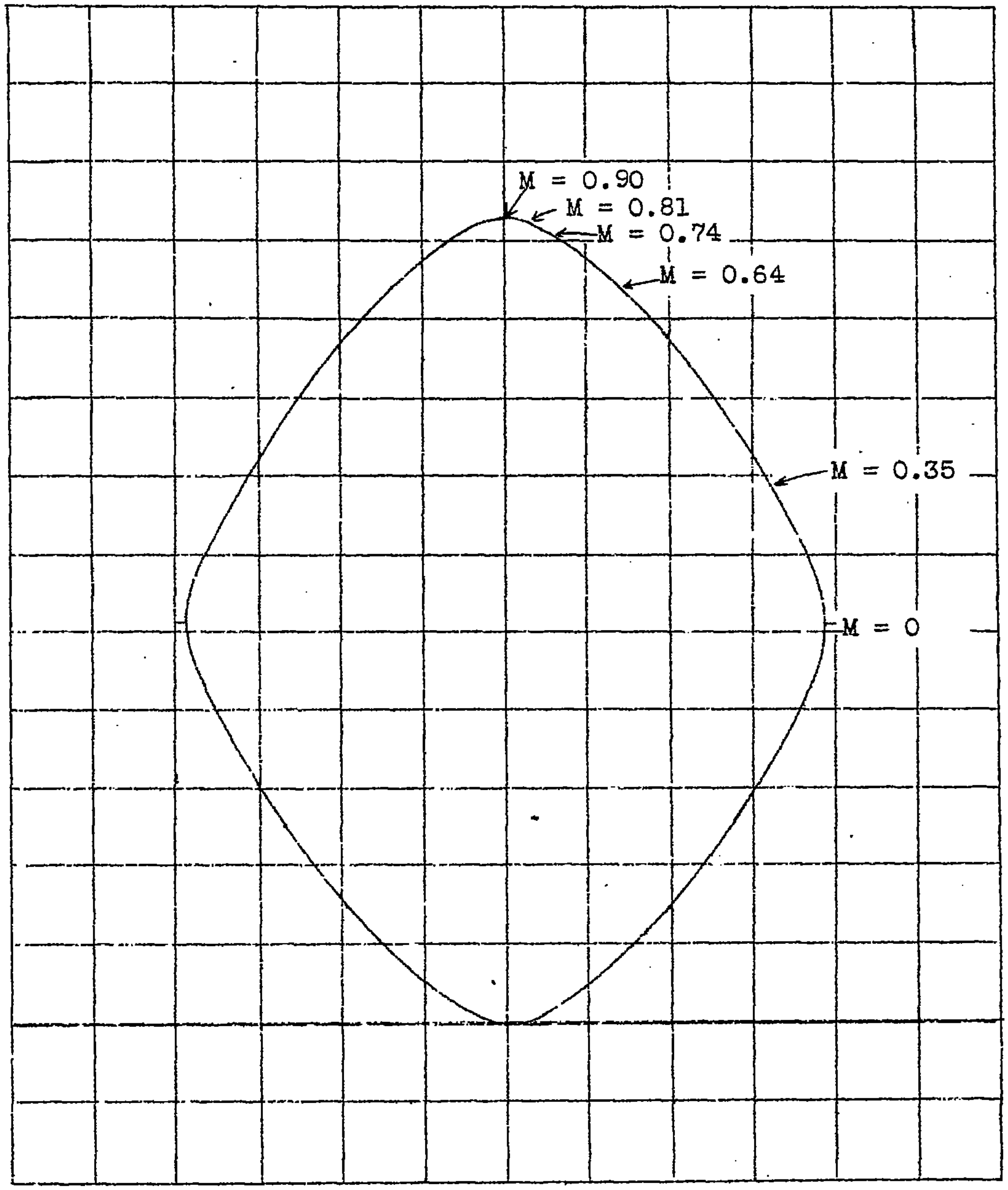

Figure 10.- The boundary, B_1, of the obstacle. $k = -0.5$, $\alpha = 0.1$

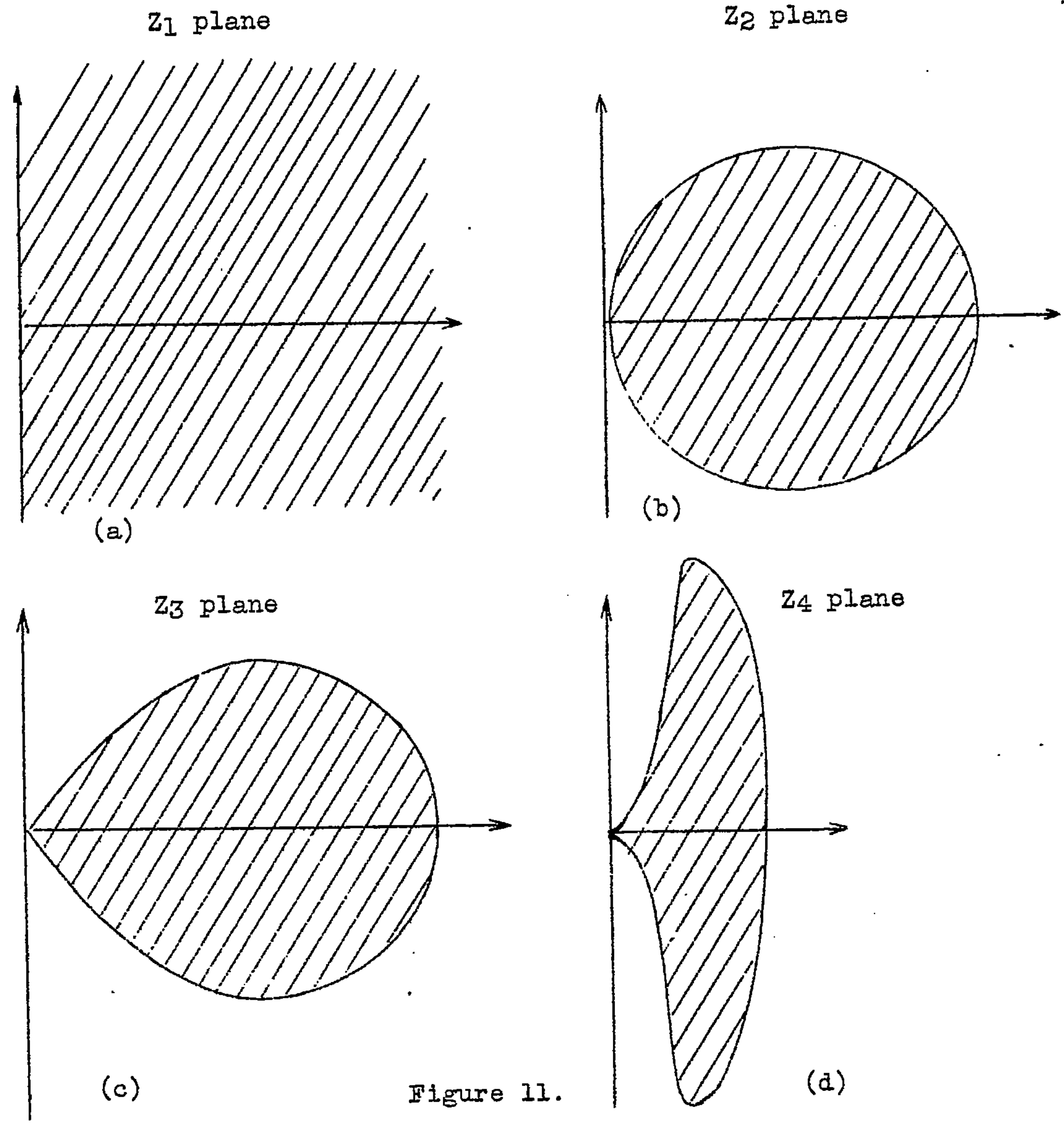

Figure 11.

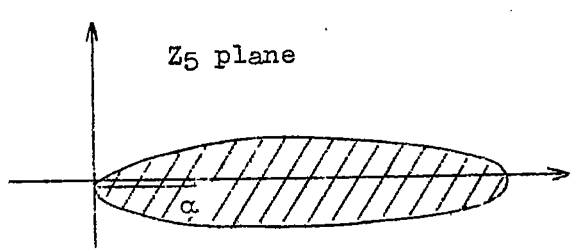

Figure 12.